MTH 516/616: Topology II
Semester 2, 2022-23

January 16, 2023

Contents

1 Homology

1.1 Simplicial Homology

(i) Motivation for homology.

(ii) n-simplices and Δ-complexes.

(iii) The free abelian group $\Delta_n(X)$ generated by the n-simplices.

(iv) The boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$, defined by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

(v) The composition $\partial_n \partial_{n-1} = 0$, and hence we have the chain complex

$$\ldots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_{n-1}} \Delta_n(X) \to \ldots \to \Delta_0(X) \xrightarrow{\partial_0} 0.$$

(vi) The simplicial homology group $H_n^{\Delta}(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$.

(vii) The simplicial homologies of S^2, $S^1 \times S^1$, $\mathbb{R}P^2$, and the Klein bottle.

1.2 Singular Homology

(i) Singular n-simplices $\sigma : \Delta^n \to X$.

(ii) The free abelian group $C_n(X)$ of singular n-chains.

(iii) The boundary map $\partial_n(\sigma) = \sum_i \sigma| \, [v_0, \ldots, \hat{v}_i, \ldots, v_n]$.

(iv) The composition $\partial_n \partial_{n-1} = 0$, and hence we have the chain complex

$$\ldots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \ldots \to C_0(X) \xrightarrow{\partial_0} 0.$$

(v) The singular homology group $H_n(X)$.

(vi) Let $X = \sqcup_\alpha X_\alpha$, where the X_α are its path components. Then

$$H_n(X) \cong \oplus_\alpha H_n(X_\alpha).$$

(vii) If X is nonempty and path-connected , then $H_0(X) \cong \mathbb{Z}$.

(viii) If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.

(ix) The augmented chain complex

$$\ldots \to C_2(X) \to C_1(X) \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is defined by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$.

(x) The reduced homology group $\widetilde{H}_n(X)$ are the homology groups associated with the augmented chain complex.

(xi) A continuous map $f : X \to Y$ induces a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

(xii) For the composed mapping $X \xrightarrow{g} Y \xrightarrow{f} Z$, we have $(fg)_* = f_* g_*$.

(xiii) If $f, g : X \to Y$ are maps such that $f \simeq g$, then $f_* = g_*$. Consequently, $(i_X)_* = i_{H_n(X)}$.

(xiv) If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$. In particular, if X is contractible, then $\widetilde{H}_n(X) = 0$ for all n.

(xv) A continuous map $f : X \to Y$ induced a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

(xvi) If $f, g : X \to Y$ are continuous maps such that $f \simeq g$, then $f_* = g_*$.

(xvii) Suppose that $f, g : X \to Y$ be continuous maps such that $f \simeq g(via\, H)$. Let $P : C_n(X) \to C_{n+1}(Y)$ be the prism operator, which is defined by

$$P(\sigma) = \sum_i F \circ (\sigma \times i_I)|[v_0 \ldots v_i, w_i, \ldots, w_n].$$

Then $\partial P + P\partial = g_\# - f_\#$.

(xviii) If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$ for all n.

(xix) Properties of exact sequences.

(xx) For a pair (X, A), the group of relative n-chains

$$C_n(X, A) = C_n(X)/C_n(A).$$

(xxi) Relative homology groups $H_n(X, A)$.

(xxii) The boundary map $\partial : H_n(X, A) \to H_{n-1}(A)$.

(xxiii) The sequence of homology groups

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots$$

is exact.

(xxiv) The sequence of reduced homology groups

$$\ldots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X, A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \to \ldots$$

is exact.

(xxv) For the pair $(D^n, \partial D^n)$,

$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z}, & \text{for } i = n \\ 0, & \text{otherwise.} \end{cases}$$

(xxvi) For the pair $(X, \{x_0\})$, where $x_0 \in X$,

$$H_n(X, \{x_0\}) \cong \widetilde{H}_n(X).$$

(xxvii) Let (X, A, B) be a triple of spaces, where $B \subset A \subset X$. Then we have the following long exact sequence of homology groups:

$$\ldots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \ldots$$

(xxviii) If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$. then $f_* = g_*$.

(xxix) (Excision Theorem) Given subspaces $Z \subset A \subset X$ such that $\overline{Z} \subset A^\circ$, then the inclusion $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \xrightarrow{i_*} H_n(X, A)$ for all n.

(xxx) Good pairs of spaces (X, A).

(xxxi) For good pairs of spaces (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism

$$q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A),$$

for all n.

(xxxii) For good pairs (X, A), $\widetilde{H}_n(X \cup CA) \cong H_n(X, A)$.

(xxxiii) For the pair $(D^n, \partial D^n)$, we have

$$H_n(D^n, \partial D^n) = \langle [i_{\Delta^n}] \rangle,$$

where i_{Δ^n} is viewed as singular a n-cycle in $C_n(D^n, \partial D^n)$.

(xxxiv) Regard S^n as a Δ-complex built from two n-simplices Δ^n_1 and Δ^n_2 with their boundaries identified. Then we have

$$H_n(S^n) = \langle [\Delta^n_1 - \Delta^n_2] \rangle,$$

where $\Delta^n_1 - \Delta^n_2$ is viewed as singular n-cycle in $C_n(S^n)$.

(xxxv) If (X, A) is a good pair of spaces, then there is an exact sequence of reduced homology groups

$$\ldots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \to \ldots,$$

where $i : A \hookrightarrow X$ is the inclusion map and $j : X \to X/A$ is the quotient map.

(xxxvi) $\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z}, & \text{for } i = n \\ 0, & \text{otherwise.} \end{cases}$

(xxxvii) (Brouwer's fixed-point theorem) Every continuous map $f : D^n \to D^n$ has a fixed point.

(xxxviii) If a CW complex X is the union of subcomplexes A and B, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

(xxxix) If a wedge sum $\bigvee_\alpha$ of spaces is formed at base points $x_\alpha \in X_\alpha$ such that each pair (X_α, x_α) is good, then the inclusions $i_\alpha : H_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induces an isomorphism

$$\oplus_\alpha (i_\alpha)_* : \oplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\vee_\alpha X_\alpha).$$

(xl) If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

(xli) (Naturality Property) If $f : (X, A) \to (Y, b)$ is a continuous map of pairs, then the diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \ldots & \longrightarrow & H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y, B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow & \ldots \end{array}$$

is commutative.

(xlii) Let X be a Δ-complex, and A a subcomplex. Then the relative homology $H_n^\Delta(X)$ can defined using the relative chains

$$\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A).$$

(xliii) For the Δ-complex pair (X, A), there exists a long exact sequence of homology groups

$$\ldots \to H_n^\Delta(A) \xrightarrow{i} H_n^\Delta(X) \xrightarrow{j} H_n^\Delta(X, A) \xrightarrow{\partial} H_{n-1}^\Delta(A) \to \ldots,$$

(xliv) Let $\phi_* : H_n^\Delta(X, A) \to H_n(X, A)$ be the canonical homomorphism induced by the chain map $\phi : \Delta_n(X, A) \to C_n(X, A)$ sending each n-simplex Δ^n of X to its characteristic map $\sigma : \Delta^n \to X$. Then ϕ_* is an isomorphism.

1.3 Cellular homology

(i) The chain group $C_n^{CW}(X) = H_n(X^n, X^{n-1})$, and the chain complex

$$\ldots C_{n+1}^{CW}(X) \xrightarrow{d_{n+1}} C_n^{CW}(X) \xrightarrow{d_n} C_{n-1}^{CW}(X) \to \ldots,$$

where

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

and

$$d_{\alpha\beta} = \deg(S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1})$$

that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point.

(ii) The cellular homology group is defined by

$$H_n^{CW}(X) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

(iii) If X is a CW complex, then:

(a) $H_k(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } k \neq n, \text{ and} \\ \bigoplus_\alpha \langle [e^n_\alpha] \rangle & \text{if } k = n. \end{cases}$

(b) $H_k(X^n) = 0$ for $k > n$.

(c) The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$, if $k < n$.

(iv) $H^{CW}_n(X) \cong H_n(X)$.

1.4 Mayer-Vietoris Sequences

(i) For a pair of subspaces $A, B \subset X$ such that $X = A^\circ \cup B^\circ$, there is an long exact sequence of the form

$$\ldots \to H_n(A \cap B) \xrightarrow{\Phi_*} H_n(A) \oplus H_n(B) \xrightarrow{\Psi_*} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \ldots \to H_0(X) \to 0,$$

which is associated with the short exact sequence

$$0 \to C_n(A \cap B) \xrightarrow{\Phi} C_n(A) \oplus C_n(B) \xrightarrow{\Psi} C_n(A + b) \to 0,$$

where $\Phi(x) = (x, -x)$, and $\Psi(x, y) = x + y$.

(ii) There exists a long exact sequence identical to the one above involving reduced homology groups.

(iii) Viewing the Klein Bottle K as the union of two Mobius bands identified along their boundaries, we have that

$$H_n(K) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}, \oplus \mathbb{Z}_2 & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

1.5 Homology with coefficients

(i) For a fixed abelian group G, the abelian chain groups $C_n(X; G) = \{\sum_i n_i \sigma_i \ : \ n_i \in G \text{ and } \sigma_i : \Delta^n \to X\}$.

(ii) The relative chain groups $C_n(X, A; G) = C_n(X; G)/C_n(A; G)$.

(iii) Both $C_n(X; G)$ and $C_n(X, A; G)$ form chain complexes, and the homology groups of their associated homology groups with coefficients in G are denoted by $H_n(X; G)$ and $H_n(X, A; G)$ respectively.

(iv) When $G = \mathbb{Z}_2$, n-chains are simply sums (or maybe viewed as unions) of finitely many singular n-simplices. Hence, this is the most natural tool in the absence of orientation.

(v) Mayer-Vietros sequence and the Cellular homology generalise to homology with coefficients.

(vi) If $f : S^k \to S^k$ has degree m, then $f_* : H_k(S^k; G) \to H_k(S^k; G)$ is multiplication by m.

(vii) Let F be a field of characteristic 2. Then

$$H_n(\mathbb{R}P^n; F) \cong \begin{cases} F, & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

(viii) Given an abelian group G and an integer $n \geq 1$, the Moore space $M(G, n)$ is a CW- complex X satisfying

  (a) $H_n(X) \cong G$ and $\widetilde{H}_i(X)$, if $i \neq n$, and

  (b) X is simply-connected if $n > 1$.

(ix) The Moore space $X = M(\mathbb{Z}_m, n)$ is obtained by attached e^{n+1} to S^n by a degree m map.

1.6 Applications of homology

(i) The degree of a map $f : S^n \to S^n$ denoted by $\deg f$, and its properties.

(ii) S^n has a continuous tangent vector field iff n is odd.

(iii) For n even, $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n.

(iv) The local degree of a map $f : S^n \to S^n$ at a point x_i denoted by $\deg f|_{x_i}$.

(v) $\deg f = \sum_i \deg f|_{x_i}$.

(vi) The map $z^k : S^1 \to S^1$ has degree k.

(vii) Constructing a map $f : S^n \to S^n$ of any given degree k.

(viii) If $Sf : S^{n+1} \to S^{n+1}$ is the suspension of the map $f : S^n \to S^n$, then $\deg Sf = \deg f$.

(ix) $H_i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$

(x) $H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0 \text{ or } i = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } i \text{ odd, } 0 < i < n \\ 0 & \text{otherwise.} \end{cases}$

(xi) Let S_g denote the closed orientable surface of genus g. Then

$$H_i(S_g) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2 \\ \mathbb{Z}^{2g} & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(xii) Let N_g denote the closed nonorientable surface with g crosscaps. Then

$$H_i(N_g) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(xiii) The Euler Characteristic of a finite-dimensional CW complex X having c_i i-cells for $0 \leq i \leq n$, is given by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_n.$$

(xiv) If $X = X^n$ is a CW complex, then

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_n(X).$$

(xv) If $X = X^n$ and $Y = Y^n$ are CW complexes such that $X \approx Y$, then $\chi(X) = \chi(Y)$.

(xvi) If $r : X \to A$ is a retraction, then $i_* H_n(A) \to H_n(X)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. Hence, we have a short exact sequence

$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \to 0$$

that splits since $r_* \circ i_* = (i_A)_*$. Consequently,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

(xvii) A $K(G, 1)$ space X is a path-connected space with contractible universal cover, and which satisfies $\pi_1(X) \cong G$.

(xviii) If a finite-dimensional CW complex is a $K(G, 1)$, then the group $G = \pi_1(X)$ is torsion-free.

(xix) If D is a subspace of S^n homeomorphic to D^k for some $k \geq 0$, then $\widetilde{H}_i(S^n - D) = 0$, for all i.

(xx) (Generalised Jordan Curve Theorem). If S is a subspace of S^n homeomorphic to S^k for some k with $0 \leq k \leq n$, then

$$\widetilde{H}_i(S^n - S) \cong \begin{cases} \mathbb{Z} & \text{for } i = n - k - 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

(xxi) (Invariance of Domain) If a subspace X of $\mathbb{R}^n$ is homeomorphic to an open set in $\mathbb{R}^n$, then X is itself open in $\mathbb{R}^n$.

(xxii) If M is a compact n-manifold and N is a connected n-manifold, then an embedding $M \hookrightarrow N$ must be surjective.

(xxiii) An odd map $f : S^n \to S^n$ must have odd degree.

(xxiv) (Borsuk-Ulam Theorem) For every map $g : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $g(x) = g(-x)$.

2 Singular Cohomology

(i) Motivation for cohmology.

(ii) The cochain complex C^* of free abelian groups

$$\ldots \leftarrow C^*_{n+1} \xleftarrow{\delta_{n+1}} C^*_n \xleftarrow{\delta_n} C^*_{n-1} \leftarrow \ldots,$$

is the dual of the chain complex C

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots,$$

where for all i, $C^*_i = \mathrm{Hom}(C_i, G)$ and $\delta_i = \partial^*_i$,

(iii) The cohomology groups

$$H^n(C; G) = \mathrm{Ker}\,\delta_{n+1}/\mathrm{Im}\,\delta_n.$$

(iv) There exists a natural map $h : H^n(C^*; G) \to \mathrm{Hom}(H_n(C), G)$, which yields the following split short exact sequence

$$0 \to \mathrm{Ker}\,h \to H^n(C^*; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0.$$

(v) There is a long exact sequence

$$\ldots \leftarrow B_n^* \overset{i_n^*}{\longleftarrow} Z_n^* \leftarrow H^n(C^*;G) \leftarrow B_{n-1}^* \leftarrow \ldots$$

associated with the short exact sequence

$$0 \leftarrow Z_n^* \overset{j_n^*}{\longleftarrow} C_n^* \overset{\delta_n}{\longleftarrow} B_{n-1}^* \leftarrow 0,$$

where i_n^* and i_* are the duals of the inclusions $i_n : B_n \hookrightarrow Z_n$, and $j_n : Z_n \hookrightarrow C_n$, respectively. This long exact sequence can be expressed as the direct sum of (or can be decomposed to) the split short exact sequences

$$0 \to \operatorname{Coker} i_{n-1}^* \to H^n(C^*;G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0.$$

(vi) A free resolution F_H of an abelian group H is an exact sequence of free groups

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

The dual of the free resolution is denoted by ${F_H}^*$.

(vii) A homomomorphism $\alpha : H \to H'$ induces a chain map from $F_H \to F_{H'}$. Furthermore, any two such chain maps are chain homotopic.

(viii) For any two free resolutions F_H and F'_H of H, there are canonical isomorphisms $H^n({F_H}^*;G) \cong H^n({F'_H}^*;G)$.

(ix) Since every abelian group H has a free resolution of the form

$$0 \to F_1 \to F_0 \to H \to 0,$$

$H^n({F_H}^*;G) = 0$, for $n > 1$, and $H^n({F_H}^*;G)$ depends only on H and G, and is denoted by $\operatorname{Ext}(H,G)$.

(x) The group $\operatorname{Ext}(H,G)$ has the following properties.

(i) $\operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H,G) \oplus \operatorname{Ext}(H',G)$.

(ii) $\text{Ext}(H, G) = 0$, if H is free.

(iii) $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

(xi) Since there is a free resolution F_H when $H = H_{n-1}(C)$

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0,$$

its dual ${F_H}^*$

$$0 \leftarrow B^*_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow H_{n-1}(C)^* \leftarrow 0$$

yields the isomorphisms

$$\text{Coker}(i^*_{n-1}) \cong \text{Ext}(H_{n-1}(C), G).$$

(xii) (Universal Coefficient Theorem for Cohomology) If C is a chain complex of free abelian groups, then the cohomology groups $H^n(C; G)$ of the cochain complex C^* are determined by the split exact sequence

$$0 \to \text{Ext}(H_{n-1}(C), G) \to H^n(C^*; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \to 0.$$

Consequently, we have the isomorphisms

$$H^n(C^*; G) \cong \text{Ext}(H_{n-1}(C), G) \oplus \text{Hom}(H_n(C), G).$$

(xiii) Let the homology groups $H_n = F_n \oplus T_n$ and $H_{n-1} = F_{n-1} \oplus T_{n-1}$ of a chain complex C be finitely generated abelian groups. Then

$$H^n(C^*; Z) \cong T_{n-1} \oplus (H_n/T_n).$$

(xiv) If a chain map between two chain complexes of free abelian groups induces an isomorphism of homology groups, then it induces isomorphisms of cohomology groups with any coefficient group G.

2.1 Cup product

(i) Let R be a commutative ring with identity. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the cup product $\varphi \smile \psi$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+\ell}]).$$

(ii) For $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$,

$$\delta(\varphi \smile \psi)(\sigma) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

(iii) The cup product has the following properties.

(a) The cup product of two cocycles is a cocycle.

(b) The cup product of a cocycle and a coboundary in any order is a coboundary.

(c) Hence the cup product induces a map at the level of cohomology

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R),$$

which is both associative and distributive.

(iv) For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

(v) For a commutative ring R with identity, $H^*(X;R) = \oplus_k H^k(X;R)$ forms a commutative ring with identity. Furthermore, $H^*(X;R)$ is a graded ring under $\smile$.

(vi) For a graded ring A with decomposition $A = \oplus_{k \geq 0} A_k$, to indicate that $a \in A$ lies in A_k, we write $|a| = k$.

(vii) $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, and $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$, where $|\alpha| = 1$. In the complex case, $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$, where $|\alpha| = 2$.

(viii) The inclusions $i_\alpha : X_\alpha \hookrightarrow \sqcup_\alpha X_\alpha$ induce the isomorphism

$$H^*(\sqcup_\alpha X_\alpha; R) \cong \prod_\alpha H^*(X_\alpha; R).$$

(ix) For basepoints $x_\alpha \in X_\alpha$, if (X_α, x_α) form good pairs, then we have that

$$\widetilde{H}^*(\vee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R).$$

(x) If R is a commutative ring, then

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha,$$

for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

(xi) $\mathbb{C}P^2$ is not homotopically equivalent to $S^2 \vee S^4$, even though they have isomorphic homology and cohomology groups.

2.2 Orientations and homology

(i) The *local orientation of an n-manifold M at a point x* is a choice of generator μ_x of the group the infinite cyclic group $H_n(M, M - \{x\}) \cong \mathbb{Z}$.

(ii) Every manifold M has a two-sheeted covering space

$$\widetilde{M} = \cup_{x \in M} \{\mu_x, \mu_{-x}\}.$$

(iii) The covering space $\widetilde{M} \to M$ can be imbedded in a larger covering space $M_{\mathbb{Z}} \to M$ given by

$$M_{\mathbb{Z}} = \cup_{x \in M} \{0, \mu_{\pm x}, \mu_{\pm 2x}, \ldots\},$$

where $\mu_{kx} \leftrightarrow k \in \mathbb{Z} \cong H_n(M|x)$.

(iv) A continuous map $M \to M_{\mathbb{Z}}$ of the form $x \mapsto \alpha_x \in H_n(M|x)$ is called a *section* of covering space.

(v) An *orientation for* M is a section such that α_x is a generator for each x. If there exists an orientation for M, then M is said to *orientable*.

(vi) An R-*orientation for* M, where R is a commutative ring with identity, is a section of the covering space M_R that assigns to each $x \in M$, a generator $\alpha_x \in H_n(M|x; R)$.

(vii) Let M be an n-manifold. Then:

(a) $\widetilde{M}$ is orientable, and

(b) if M is connected, then M is orientable if, and only if $\widetilde{M}$ has two components. In particular, M is orientable, if its simply-connected, or more generally, if $\pi_1(M)$ has no subgroup of index 2.

(viii) An orientable manifold is R-orientable for all R, while a nonorientable manifold is R-orientable if, and only if R contains a unit of order 2. In particular, every manifold is $\mathbb{Z}_2$-orientable.

(ix) Let M be a manifold of dimension n, and let $A \subset M$ be a compact subset. Then:

(a) $H_i(M|A; R) = 0$ for $i > n$, and a class in $H_n(M|A; R)$ is zero if, and only if its image in $H_n(M|x; R)$ is zero for all $x \in A$.

(b) If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there exists a unique class $\alpha_A \in H_n(M|A; R)$ whose image in $H_n(M|x; R)$ is a α_x for all $x \in A$.

(x) Let M be a closed connected n-manifold, Then:

(a) If M is R-orientable, the map $H_n(M; R) \to H_n(M|x; R) \cong R$ is an isomorphism for all $x \in M$.

(b) If M is not R-orientable, the map $H_n(M; R) \to H_n(M|x; R) \cong R$ is injective with image $\{r \in R \,|\, 2r = 0\}$ for all $x \in M$.

(c) $H_i(M; R) = 0$, for $i > n$.

(xi) An element $[M] \in H_n(M;R)$ whose image in $H_n(M|x;R)$ is a generator for all x is called a *fundamental class.*

(xii) If M is a closed connected n-manifold, the torsion subgroup of $H_{n-1}(M;Z)$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is orientable.

2.3 Cap product and Poincaré Duality

(i) For an arbitrary space X and a coefficient ring R, we define an R-linear *cap product* map

$$\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

for $k \geq \ell$, by sending a singular k-simplex $\sigma : \Delta^k \to X$ and a cochain $\varphi \in C^\ell(X;R)$ to the singular $(k-\ell)$-simplex

$$\sigma \frown \varphi = \varphi(\sigma|[v_0, \ldots, v_\ell])\sigma|[v_\ell, \ldots, v_k].$$

(ii) The cap product has the following properties:

(a) For any $\sigma \in C_k(X;R)$ and $\varphi \in C^\ell(X;R)$,

$$\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi).$$

(b) Cap product of a cycle and a cocycle is a cocycle.

(c) Cap product of a cycle and a coboundary is a boundary.

(d) Cap product of a boundary and a cocycle is a boundary.

(e) Thus, there is an induced cap product

$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$

that is R-linear in each variable.

(f) Given a map $f : X \to Y$,

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi)).$$

(iii) (Poincaré Duality for closed manifolds) If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D : H^k(M;r) \to H_{n-k}(M;r)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.

(iv) Let $C_c^i(X;G)$ be the subgroup of $C^i(X;G)$ consisting of all cochains $\varphi : C_i(X) \to G$ that are supported by a compact subset $K_\varphi \subset X$. The cohomology groups $H_c^i(X;G)$ of this subcomplex are called *cohomology groups with compact support.*

(v) Let $X_c = \{K \subset X \,|\, K \text{ is compact}\}$, then

$$C_c^i(X;G) = \bigcup_{K \in X_c} C^i(X, X-K; G).$$

(vi) For $K, L \in X_c$ such that $K \subset L$, the inclusion $K \hookrightarrow L$ induces inclusions $C^i(X, X-K; G) \hookrightarrow C^i(X, X-L; G)$.

(vii) Consequently, $\{H^i(X, X-K; G) \,|\, K \in X_c\}$ forms a directed system of groups, and we have

$$H_c^i(X;G) = \varinjlim_{K \in X_c} H^i(X, X-K; G).$$

(viii) Suppose that $X = \cup_{\alpha \in J} X_\alpha$, where J is a directed set. If for each compact $K \subset X$, there exists $\alpha = \alpha(K) \in J$ such that $K \subset X_\alpha$, then we have

$$H_i(X;G) \cong \varinjlim H_i(X_\alpha;G).$$

(ix)

$$H_c^i(\mathbb{R}^n;G) \cong \begin{cases} G, & \text{for } i = n, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

3 Homotopy Groups

(i) For a pair (X, x_0), we define

$$\pi_n(X, x_0) := \{[f] \mid f : (I^n, \partial I^n) \to (X, x_0)\}.$$

(ii) Alternatively, we can define

$$\pi_n(X, x_0) := \{[f] \mid f : (S^n, s_0) \to (X, x_0)\}.$$

(iii) When $n \geq 2$, we define an operation $+$ in $\pi_n(X, x_0)$ by:

$$(f+g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \ldots, s_n), & s_1 \in [1/2, 1], \end{cases}$$

and $[f] + [g] := [f + g]$.

(iv) $(\pi_n(X, x_0), +)$ is an abelian group.

(v) Let X be a path-connected space. Given a path $\gamma : I \to X$ from x_0 to x_1, we can associate to each $f : (I^n, \partial I^n) \to (X, x_0)$ a map $f_\gamma : (I^n, \partial I^n) \to (X, x_1)$ satisfying the following properties

(a) $(f + g)_\gamma \simeq f_\gamma + g_\gamma$.

(b) $f_{\gamma\eta} \simeq (f_\eta)_\gamma$.

(c) $f_e \simeq f$, where $e = e_{x_0}$.

(vi) Hence there is an induced homomorphism

$$\Phi_\gamma : (\pi_n(X, x_1), +) \to (\pi_n(X, x_0), +)$$

given by $\Phi([f]) = [f_\gamma]$ which is an isomoprhism.

(vii) A covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0)$ for all $n \geq 2$. Consequently, $\pi_n(X, x_0) = 0$ for $n \geq 2$ whenever X has a contractible universal cover.

(viii) Let $(X_\alpha, x_\alpha)_{\alpha\in J}$ be an arbitrary collection path-connected spaces. Then the projection maps $p_\alpha : \prod_{\beta\in J}(X_\beta, x_\beta) \to (X_\alpha, x_\alpha)$ induces the isomorphism

$$\pi_n(\prod_{\alpha\in J}(X_\alpha, x_\alpha)) \cong \prod_{\alpha\in J}\pi_n(X_\alpha, x_\alpha).$$

(ix) For a pair of spaces (X, A) with a basepoint $x_0 \in A$ and $n \geq 1$, the *relative homotopy groups* $(\pi_n(X, A, x_0), +)$ are defined by

$$\pi_n(X, A, x_0) = \{[f] \mid f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\},$$

where $J^{n-1} = \overline{\partial I^n - I^{n-1}}$. Alternatively, it is defined by

$$\pi_n(X, A, x_0)_=\{[f] \mid f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)\},$$

where the addition is done via the map $c : D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

(x) A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if, and only if it is homotopic rel S^{n-1} to a map with image contained in A.

(xi) For a pair of spaces (X, A) with a basepoint $x_0 \in A$, the sequence

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots$$

is exact.

(xii) For a triple of spaces (X, A, B) with $B \subset A \subset X$ and a basepoint $x_0 \in B$, the sequence

$$\ldots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \ldots$$

is exact.

(xiii) (Whitehead Theorem) Suppose that a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n. Then:

(a) f is homotopically equivalent to Y, and

(b) furthermore if X is a subcomplex of Y, then X is a deformation retract of Y.